

ON VANISHING OF UNRAMIFIED COHOMOLOGY OF GEOMETRICALLY RATIONAL VARIETIES OVER FINITE FIELDS

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ABSTRACT. The purpose of this note is to show that the third unramified cohomology $H_{Zar}^0(X, \mathcal{H}^3(\mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))$ of a smooth projective geometrically rational variety X of dimension 3 over a finite field $k = \mathbb{F}_q$ must vanish under \mathbb{Z}_ℓ -exactness Hard Lefschetz condition.

1. INTRODUCTION

Let k be a field and $\ell \neq \text{char}(k) = p$ be any prime. Let X be a smooth projective geometrically integral k -variety. Denote by $\mathcal{H}_{\acute{e}t}^n(\mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ resp. $\mathcal{H}_{\acute{e}t}^n(\mu_{\ell^m}^{\otimes j})$ the Zariski sheaf on X associated to the presheaf $U \mapsto H_{\acute{e}t}^n(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ resp. $U \mapsto H_{\acute{e}t}^n(U, \mu_{\ell^m}^{\otimes j})$. If $F = k(X)$ is the function field of X , then we write $H_{nr}^n(F/k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ resp. $H_{nr}^n(F/k, \mu_{\ell^m}^{\otimes j})$ for the unramified cohomology with ℓ -divisible resp. finite coefficients. We denote by $\alpha : X_{\acute{e}t} \rightarrow X_{Zar}$ the obvious morphism. In fact, one has $\mathcal{H}_{\acute{e}t}^n(A(j)) = R^n \alpha_* A(j)$. By a geometrically rational variety over a field k we mean a smooth projective variety X such that $\overline{X} = X \otimes_k \overline{k}$ is a rational variety. For a smooth projective variety Y of dimension $n + 1$ over $\overline{\mathbb{F}}_q$ with a smooth hyperplane section Z , we say that Y satisfies \mathbb{Z}_ℓ -exactness Hard Lefschetz condition if one has a direct decomposition

$$(1.1) \quad H_{\acute{e}t}^n(Z, \mathbb{Z}_\ell) = H_{\acute{e}t}^n(Z, \mathbb{Z}_\ell)_{ev} \oplus H_{\acute{e}t}^n(Y, \mathbb{Z}_\ell)$$

where we choose an isomorphism $\mathbb{Z}_\ell \simeq \mathbb{Z}_\ell(1)$ and forget about Tate-twist and $H_{\acute{e}t}^n(Z, \mathbb{Z}_\ell)_{ev}$ denotes the space of vanishing cycles of $H_{\acute{e}t}^n(Z, \mathbb{Z}_\ell)$. Our main result is the following theorem:

Theorem 1.0.1. *Let X be a smooth projective geometrically rational variety of dimension 3 over a finite field $k = \mathbb{F}_q$ with function field $F = k(X)$ and $\ell \neq \text{char}(k) = p$ be a prime such that \overline{X} satisfies the \mathbb{Z}_ℓ -exactness Hard Lefschetz condition, then the third unramified cohomology $H_{nr}^3(F/k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is trivial.*

The \mathbb{Z}_ℓ -exactness Hard Lefschetz condition is in fact the question in [CTK11, Ques. 5.7], which we certainly can not answer in this note.

2. PROOF OF THEOREM 1.0.1

In this section we prove the main theorem 1.0.1 through several steps. First of all we show

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Proposition 2.0.2. *Let X be a smooth projective geometrically integral variety over a field k of characteristic $\text{char}(k) \geq 0$ and $\ell \neq \text{char}(k)$ be a prime. Then one has an exact sequence*

$$(2.2) \quad 0 \rightarrow \text{CH}^1(X) \otimes \mathbb{Z}_\ell \rightarrow H_{\text{ét}}^2(X, \mathbb{Z}_\ell(2)) \rightarrow \varprojlim_n H_{\text{nr}}^2(F/k, \mu_{\ell^n})$$

Moreover, the group $\varprojlim_n H_{\text{nr}}^2(F/k, \mu_{\ell^n})$ is torsion-free.

Proof. By Kummer theory one has a distinguished triangle, see [Voe03, Thm. 6.6]

$$\mathbb{Z}/\ell^n(1) \rightarrow R\alpha_* \alpha^* \mathbb{Z}/\ell^n(1) \rightarrow \tau_{\geq 2} R\alpha_* \alpha^* \mathbb{Z}/\ell^n(1) \xrightarrow{+1}$$

By taking cohomology we have an exact sequence

$$0 \rightarrow \text{CH}^1(X) \otimes \mathbb{Z}/\ell^n \rightarrow H_{\text{ét}}^2(X, \mu_{\ell^n}) \rightarrow \mathbb{H}_{\text{Zar}}^2(X, \tau_{\geq 2} R\alpha_* \alpha^* \mathbb{Z}/\ell^n(1)) \rightarrow 0$$

One has a spectral sequence [SV00, Thm. 0.3]

$$E_2^{p,q} = H_{\text{Zar}}^p(X, \underline{H}^q(\tau_{\geq 2} R\alpha_* \alpha^* \mathbb{Z}/\ell^n(1))) \Rightarrow \mathbb{H}_{\text{Zar}}^{p+q}(X, \tau_{\geq 2} R\alpha_* \alpha^* \mathbb{Z}/\ell^n(1)),$$

where \underline{H}^q denote the cohomology sheaves. By the exact sequence for terms of lower degree one has an injection

$$0 \rightarrow \mathbb{H}_{\text{Zar}}^2(X, \tau_{\geq 2} R\alpha_* \alpha^* \mathbb{Z}/\ell^n(1)) \rightarrow H_{\text{nr}}^2(F/k, \mu_{\ell^n})$$

Since there is no differentials for $E_r^{0,2}$, we have $H_{\text{nr}}^2(F/k, \mu_{\ell^n}) \stackrel{\text{def}}{=} E_2^{0,2} = E_\infty^{0,2}$. So the injection above is in fact an isomorphism, so it gives us the exact sequence 2.2. Now by definition we have $\varprojlim_n H_{\text{nr}}^2(F/k, \mu_{\ell^n}) \subset H_{\text{ét}}^2(F, \mathbb{Z}_\ell(1))$. The last group is torsion-free by Kummer theory, so we are done. \square

Proposition 2.0.3. *Let X be a smooth projective geometrically integral variety of dimension d over a field k with the function field $F = k(X)$. Let $k \subset \Omega$ be a universal domain in sense of Weil. Assume $\text{CH}_0(X_\Omega) = \mathbb{Z}$, then $H_{\text{nr}}^p(F/k, \mu_{\ell^n}^{\otimes j})$ are killed by an integer $N \geq 1$, for all $p > cd_\ell(k)$.*

Proof. The assumption that $\text{CH}_0(X_\Omega) = \mathbb{Z}$ implies the diagonal decomposition in $\text{CH}^d(X \times X)$ (see [BS83])

$$N\Delta_X = \Gamma_1 + \Gamma_2,$$

where Γ_1 is supported on $\xi \times X$ with ξ is a 0-dimensional subscheme, Γ_2 is supported on $X \times D$ for a divisor $D \subset X$ and $N \in \mathbb{N}^\times$ is an integer. By action of correspondences, see e.g. [CTV10, App.], we obtain

$$NId = \Gamma_{1*} + \Gamma_{2*} : H_{\text{Zar}}^0(X, \mathcal{H}_{\text{ét}}^p(\mu_{\ell^n}^{\otimes j})) \rightarrow H_{\text{Zar}}^0(X, \mathcal{H}_{\text{ét}}^p(\mu_{\ell^n}^{\otimes j})).$$

One has that Γ_{1*} factors through

$$H_{\text{Zar}}^0(X, \mathcal{H}_{\text{ét}}^p(\mu_{\ell^n}^{\otimes j})) \rightarrow H_{\text{Zar}}^0(\xi, \mathcal{H}_{\text{ét}}^p(\mu_{\ell^n}^{\otimes j})),$$

where we can assume ξ is a closed point and so $H_{\text{Zar}}^0(\xi, \mathcal{H}_{\text{ét}}^p(\mu_{\ell^n}^{\otimes j}))$ is trivial for $p > cd_\ell(k)$. One has that $\Gamma_{2*} = 0$, since Γ_2 is supported on $D \subsetneq X$. This shows that $H_{\text{Zar}}^0(X, \mathcal{H}_{\text{ét}}^p(\mu_{\ell^n}^{\otimes j}))$ are killed by an integer $N \geq 1$ for all $p > cd_\ell(k)$. \square

Proposition 2.0.4. *Let X be a smooth projective geometrically rational variety of dimension 3 over a finite field \mathbb{F}_q , then $H_{\text{ét}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$. In particular, $H_{\text{ét}}^4(X, \mathbb{Z}_\ell(2))$ is torsion-free.*

Proof. Let $G = \text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ be the absolute Galois group of \mathbb{F}_q . From the Hochschild-Serre spectral sequence

$$E_2^{a,b} = H_{Gal}^a(\mathbb{F}_q, H_{\acute{e}t}^b(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \Rightarrow H_{\acute{e}t}^{a+b}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

one has a short exact sequence

$$0 \rightarrow H_{Gal}^1(\mathbb{F}_q, H_{\acute{e}t}^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \rightarrow H_{\acute{e}t}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \rightarrow H_{\acute{e}t}^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^G \rightarrow 0$$

From the universal coefficient exact sequence

$$(2.3) \quad 0 \rightarrow H_{\acute{e}t}^2(\overline{X}, \mathbb{Z}_\ell(2)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow H_{\acute{e}t}^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \rightarrow H_{\acute{e}t}^3(\overline{X}, \mathbb{Z}_\ell(2))_{tors} \rightarrow 0$$

and from the fact by Serre, see e.g. [A-M], that $H_{\acute{e}t}^3(\overline{X}, \mathbb{Z}_\ell(2))$ is torsion-free, we see that $H_{\acute{e}t}^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is divisible. Since \mathbb{F}_q has cohomological dimension 1, it implies $H_{Gal}^1(\mathbb{F}_q, H_{\acute{e}t}^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))$ is also divisible. By Weil conjecture, see e.g. [CTSS83], the group $H_{\acute{e}t}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is finite, so $H_{Gal}^1(\mathbb{F}_q, H_{\acute{e}t}^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))$ is trivial and we must have

$$H_{\acute{e}t}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \cong H_{\acute{e}t}^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^G$$

So it is enough to show that $H_{\acute{e}t}^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$. By universal coefficient exact sequence

$$0 \rightarrow H_{\acute{e}t}^3(\overline{X}, \mathbb{Z}_\ell(2)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow H_{\acute{e}t}^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \rightarrow H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2))_{tors} \rightarrow 0,$$

and the fact by Serre that $H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2))$ is torsion-free, we conclude that

$$H_{\acute{e}t}^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = H_{\acute{e}t}^3(\overline{X}, \mathbb{Z}_\ell(2)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

The last group is by [Kah11, Cor. 4.20] isomorphic to the torsion subgroup of the kernel of the map

$$\mathbb{H}_{\acute{e}t}^4(\overline{X}, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell \rightarrow H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2)),$$

where we denote by $\mathbb{H}_{\acute{e}t}^n(-, \mathbb{Z}(j))$ the étale motivic cohomology. Consider the cycle class map

$$(2.4) \quad cl_X^2 : \text{CH}^2(\overline{X}) \otimes \mathbb{Z}_\ell \rightarrow \mathbb{H}_{\acute{e}t}^4(\overline{X}, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell \rightarrow H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2))$$

From the Bloch-Ogus spectral sequence [BO74]

$$E_2^{i,j} = H_{Zar}^i(\overline{X}, \mathcal{H}_{\acute{e}t}^j(\mathbb{Z}_\ell(2))) \Rightarrow H_{\acute{e}t}^{i+j}(\overline{X}, \mathbb{Z}_\ell(2))$$

one has an exact sequence

$$\begin{aligned} 0 \rightarrow N^1 H_{\acute{e}t}^3(\overline{X}, \mathbb{Z}_\ell(2)) \rightarrow H_{\acute{e}t}^3(\overline{X}, \mathbb{Z}_\ell(2)) \rightarrow H_{Zar}^0(\overline{X}, \mathcal{H}_{\acute{e}t}^3(\mathbb{Z}_\ell(2))) \rightarrow \\ \rightarrow \text{CH}^2(\overline{X}) \otimes \mathbb{Z}_\ell \xrightarrow{cl_X^2} H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2)), \end{aligned}$$

where N^1 is the first step coniveau filtration. Since $H_{Zar}^0(\overline{X}, \mathcal{H}_{\acute{e}t}^3(\mathbb{Z}_\ell(2))) = 0$, we get the injectivity of cl_X^2 . Moreover, from [Kah11, Prop. 2.8] one has an exact sequence

$$0 \rightarrow \text{CH}^2(\overline{X}) \otimes \mathbb{Z}_\ell \rightarrow \mathbb{H}_{\acute{e}t}^4(\overline{X}, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell \rightarrow H_{Zar}^0(\overline{X}, \mathcal{H}_{\acute{e}t}^3(\mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \rightarrow 0$$

Since $H_{Zar}^0(\overline{X}, \mathcal{H}_{\acute{e}t}^3(\mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) = 0$, we have an isomorphism

$$\text{CH}^2(\overline{X}) \otimes \mathbb{Z}_\ell \cong \mathbb{H}_{\acute{e}t}^4(\overline{X}, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell.$$

Apply now the Kernel-Cokernel exact sequence for the composition 2.4, we can conclude that $H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell$ maps injectively to $H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2))$. So $H_{\acute{e}t}^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is trivial. Now from the exact sequence

$$\cdots \rightarrow H_{\acute{e}t}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \rightarrow H_{\acute{e}t}^4(X, \mathbb{Z}_\ell(2)) \rightarrow H_{\acute{e}t}^4(X, \mathbb{Q}_\ell) \rightarrow \cdots$$

we see that $H_{\acute{e}t}^4(X, \mathbb{Z}_\ell(2))$ is torsion-free. \square

Proposition 2.0.5. *Let X be a smooth projective geometrically rational variety of dimension 3 over a finite field \mathbb{F}_q . Assume that \overline{X} satisfies the condition 1.1, then the cycle class map*

$$cl_X^2 : CH^2(X) \otimes \mathbb{Z}_\ell \rightarrow H_{\acute{e}t}^4(X, \mathbb{Z}_\ell(2))$$

is surjective.

Proof. X is geometrically rational, we have the base change condition $CH_0(X_\Omega) = \mathbb{Z}$. So by 2.0.2 and 2.0.3, we have a surjection $cl_X^1 : CH^1(X) \otimes \mathbb{Z}_\ell \twoheadrightarrow H^2(X, \mathbb{Z}_\ell(1))$. Let H be a smooth hyperplane section (over \mathbb{F}_q see [Poo04]) and $G = \text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. Consider the commutative diagram

$$(2.5) \quad \begin{array}{ccccc} CH^1(X) \otimes \mathbb{Z}_\ell & \xrightarrow{\cong} & H_{\acute{e}t}^2(X, \mathbb{Z}_\ell(1)) & \longrightarrow & H_{\acute{e}t}^2(\overline{X}, \mathbb{Z}_\ell(1))^G \\ \downarrow -\cap H & & \downarrow -\cap H & & \downarrow -\cap \overline{H} \\ CH^2(X) \otimes \mathbb{Z}_\ell & \xrightarrow{cl_X^2} & H_{\acute{e}t}^4(X, \mathbb{Z}_\ell(2)) & \longrightarrow & H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2))^G \end{array}$$

Since $H_{\acute{e}t}^2(\overline{X}, \mathbb{Z}_\ell(1))$ and $H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2))$ are torsion-free by Serre, see e.g. [A-M], the G -equivariant map

$$-\cap \overline{H} : H_{\acute{e}t}^2(\overline{X}, \mathbb{Z}_\ell(1)) \rightarrow H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2))$$

is then an isomorphism under our assumption 1.1 by Hard Lefschetz theorem [Del80, Thm. 4.1.1] (see [Del80, p. 223] for \mathbb{Z}_ℓ -cohomology). From the commutative diagram 2.5 we can conclude that $CH^2(X) \otimes \mathbb{Z}_\ell \rightarrow H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2))^G$ is surjective. Over a finite field \mathbb{F}_q , the Hochschild-Serre spectral sequence

$$E_2^{a,b} = H_{Gal}^a(\mathbb{F}_q, H_{\acute{e}t}^b(\overline{X}, \mathbb{Z}_\ell(2))) \Rightarrow H_{\acute{e}t}^{a+b}(X, \mathbb{Z}_\ell(2))$$

breaks up into short exact sequence

$$0 \rightarrow H_{Gal}^1(\mathbb{F}_q, H_{\acute{e}t}^3(\overline{X}, \mathbb{Z}_\ell(2))) \rightarrow H_{\acute{e}t}^4(X, \mathbb{Z}_\ell(2)) \rightarrow H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2))^G \rightarrow 0.$$

Apply now the Kernel-Cokernel exact sequence for the bottom maps of the commutative diagram 2.5 above, we have an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ker}(cl_X^2) \rightarrow \text{Ker}(CH^2(X) \otimes \mathbb{Z}_\ell \rightarrow H_{\acute{e}t}^4(\overline{X}, \mathbb{Z}_\ell(2))^G) \rightarrow \\ \rightarrow H_{Gal}^3(\mathbb{F}_q, H_{\acute{e}t}^3(\overline{X}, \mathbb{Z}_\ell(2))) \rightarrow \text{Coker}(cl_X^2) \rightarrow 0 \end{aligned}$$

By Weil conjecture, see e.g. [CTSS83], $H_{Gal}^1(\mathbb{F}_q, H_{\acute{e}t}^3(\overline{X}, \mathbb{Z}_\ell(2)))$ is finite, but from 2.0.4 we have $H_{\acute{e}t}^4(X, \mathbb{Z}_\ell(2))$ is torsion-free, so $H_{Gal}^1(\mathbb{F}_q, H_{\acute{e}t}^3(\overline{X}, \mathbb{Z}_\ell(2)))$ must vanish, hence $\text{Coker}(cl_X^2) = 0$. \square

Remark 2.0.6. In fact, the cycle class map cl_X^2 is an isomorphism for X a smooth projective geometrically rational threefold over a finite field \mathbb{F}_q under condition 1.1. The surjectivity is proved above in 2.0.5 under the condition 1.1. The injectivity follows only from the fact that $H_{Zar}^0(X, \mathcal{H}_{\acute{e}t}^i(\mathbb{Z}_\ell(2)))$ is torsion-free by Merkurjev-Suslin

theorem ($H_{\acute{e}t}^3(F, \mathbb{Z}_\ell(2))$ is torsion-free) hence it must vanish by 2.0.3 and from the exact sequence of Bloch-Ogus spectral sequence [BO74]

$$0 \rightarrow N^1 H_{\acute{e}t}^3(X, \mathbb{Z}_\ell(2)) \rightarrow H_{\acute{e}t}^3(X, \mathbb{Z}_\ell(2)) \rightarrow H_{Zar}^0(X, \mathcal{H}_{\acute{e}t}^3(\mathbb{Z}_\ell(2))) \rightarrow \\ \rightarrow \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \rightarrow H_{\acute{e}t}^4(X, \mathbb{Z}_\ell(2)),$$

without condition 1.1.

Now we use the following theorem of B. Kahn

Theorem 2.0.7. [Kah11, Thm. 1.1] *Let X be a smooth projective variety over a field k and $\ell \neq \mathrm{char}(k)$ be a prime. One has an exact sequence*

$$(2.6) \quad 0 \rightarrow H_{Zar}^0(X, \mathcal{H}_{\acute{e}t}^3(\mathbb{Z}_\ell(2))) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow H_{Zar}^0(X, \mathcal{H}_{\acute{e}t}^3(\mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \rightarrow C_{tors} \rightarrow 0,$$

where C_{tors} is the torsion subgroup of the cokernel of cl_X^2 .

As $C_{tors} = 0$ by 2.0.5, so $H_{nr}^3(F/k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is divisible by 2.6, so it must vanish by 2.0.3, so we finish the proof of the theorem 1.0.1.

Remark 2.0.8. Let X be a smooth projective threefold over an algebraic closure $\overline{\mathbb{F}}$ of a finite field \mathbb{F}_q with a smooth ample divisor $Y \hookrightarrow X$. If the Brauer group $\mathrm{Br}(Y)$ is finite, then $\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell$ maps surjectively onto $H_{\acute{e}t}^4(X, \mathbb{Z}_\ell(2))$. Indeed, $\varprojlim_n H_{nr}^2(\overline{\mathbb{F}}(Y)/\overline{\mathbb{F}}, \mu_{\ell^n})$

will be trivial under the assumption of finiteness of $\mathrm{Br}(Y)$. So by 2.0.2, we have $\mathrm{CH}^1(Y) \twoheadrightarrow H_{\acute{e}t}^2(Y, \mathbb{Z}_\ell(1))$. By weak Lefschetz theorem [Del80] one has a surjection $H_{\acute{e}t}^2(Y, \mathbb{Z}_\ell(1)) \twoheadrightarrow H_{\acute{e}t}^4(X, \mathbb{Z}_\ell(2))$. So $H_{\acute{e}t}^4(X, \mathbb{Z}_\ell(2))$ is generated by 1-cycles as one looks at the following commutative diagram

$$\begin{array}{ccccc} \mathrm{CH}^1(Y) \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{CH}^2(X - Y) \otimes \mathbb{Z}_\ell \longrightarrow 0 \\ \downarrow & & \downarrow & & \\ H_{\acute{e}t}^2(Y, \mathbb{Z}_\ell(1)) & \longrightarrow & H_{\acute{e}t}^4(X, \mathbb{Z}_\ell(2)) & & \end{array}$$

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REFERENCES

- [A-M] M. Artin, D. Mumford, Some elementary examples of unirational varieties which are not rational, Proc. Lond. Math. Soc. **25**, 75-95 (1972).
- [BO74] S. Bloch, A. Ogus, Gersten's conjecture and the homology of schemes, Ann. Sci. Sup., 4 sér **7** (1974), 181-202.
- [BS83] S. Bloch, V. Srinivas, Remark on correspondences and algebraic cycles, Amer. J. Math. **105** (1983).
- [CTSS83] J.-L. Colliot-Thélène, J.-J. Sansuc, C. Soulé, Torsion dans le groupe de Chow de codimension deux, Duke Math. J. **50** 768-801, (1983).
- [CTK11] J.-L. Colliot-Thélène, B. Kahn, Cycles de codimension 2 et H^3 non ramifié pour les variétés sur les corps finis, **arXiv : 1104.3350v1[math.AG]**
- [CTV10] J.-L. Colliot-Thélène, C. Voisin, Cohomologie non ramifiée et conjecture de Hodge entière, **arXiv : 1005.2778v1[math.AG]**
- [Del80] P. Deligne, La conjecture de Weil, II, Pub. Math. IHÉS **52** (1980), 137-252.
- [Kah11] B. Kahn, Classes de cycles motiviques étales, **arXiv : 1102.0375v2[math.AG]**
- [Poo04] B. Poonen, Bertini theorems over finite fields, Ann. of Math. **160** (2004), 1099-1127.

- [SV00] A. Suslin, V. Voevodsky, Bloch-Kato conjecture and motivic cohomology with finite coefficients, in The arithmetic and geometry of algebraic cycles, 117-189, Kluwer, (2000).
- [Voe03] V. Voevodsky, Motivic cohomology with $\mathbb{Z}/2$ coefficients, Publ. Math. IHÉS **98** (2003), 59-104.

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